Math 2010 B Tutorial 10

HW & Midterm exam $\Longrightarrow$

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Outline :

- <u>Implicit differentiation</u>

- Partial derivatives w/ constrained variables

[Ref : Section 14.10 , "Thomas" Calculus]

✩ Fact : Under mild continuity restrictions, it is true that if

$$F(x) = \int_a^b g(t, x)\, dt ,$$

then $$F'(x) = \int_a^b g_x(t, x)\, dt .$$

Challenge :

Using this fact, find the derivative of

$$\boxed{F(x) = \int_0^{x^2} \sin(xt)\, dt .} \rightsquigarrow F'(x)$$

$g(t, x) = \sin xt$ $\qquad g_x = (\cos xt) \cdot t$

Sol : Define $G(y, x) = \int_0^{\boxed{y}} \sin(xt)\, dt$

At this moment x, y are regarded as independent variables.

$\boxed{\frac{\partial G}{\partial y}} = \sin(xy) \longleftarrow$ Fund thm of Calculus

$\boxed{\frac{\partial G}{\partial x}} = \int_0^y t \cos(xt)\, dt$ (Differentiation under $\int$ sign).

Note that $F(x) = G(y = x^2, x)$. $\Rightarrow$ $F'(x) = \boxed{\frac{\partial G}{\partial y}} \cdot \underset{\underset{2x}{\parallel}}{\boxed{\frac{\partial y}{\partial x}}} + \boxed{\frac{\partial G}{\partial x}} \cdot \underset{\underset{1}{\parallel}}{\boxed{\frac{\partial x}{\partial x}}}$ w/ $y = x^2$

by Chain Rule

$$F'(x) = \frac{\partial G}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial G}{\partial x} \frac{dx}{dx} \qquad w/ \quad y = x^2$$

$$= \underset{y=x^2}{\sin(xy)} \cdot 2x + \boxed{\int_0^y t\cos(xt)\,dt} \quad \checkmark$$

$$\int_0^y t\cos(xt)\,dt = \int_0^{x^2} t\cos(xt)\,dt = \int_0^{x^2} t \cdot \frac{1}{x}\, d\sin(xt)$$

$\int$ by parts we get $\Rightarrow$

$$= \frac{t}{x}\sin(xt)\Big|_0^{x^2} - \int_0^{x^2} \underline{\frac{1}{x}\sin(xt)}\,dt$$

$$= x\sin x^3 - \left(\frac{1}{x^2} \cdot (-\cos(xt))\right)\Big|_0^{x^2}$$

$$= x\sin x^3 + \frac{\cos x^3}{x^2} - \frac{1}{x^2}$$

$$\therefore F'(x) = 3x\sin x^3 + \frac{1}{x^2}(\cos x^3 - 1)$$

Partial derivatives w/ constrained variables.

Ex 1 $\quad \underline{w = x^2 + y^2 + z^2} \qquad \dfrac{\partial w}{\partial x} \overset{?}{=} 2x$

Ex 2 : $\quad w = x^2 + y^2 + z^2$ and $\underline{z = x^2 + y^2}$ Find $\dfrac{\partial w}{\partial x}$? $\qquad \dfrac{\partial w}{\partial x} \neq 2x$

[Type of Problem

Find $\dfrac{\partial w}{\partial x_i}$ when the variables in $w = f(x_1, \cdots x_n)$ are constrained by

an equ: $\quad F(x_1, \cdots, x_n) = 0$]

In Ex 2: $\quad f(x, y, z) = x^2 + y^2 + z^2$, & $\quad F(x, y, z) = x^2 + y^2 - z = 0$

Sol of Ex 2:

[Resort :

Treat one of the variables apart from x in equ $\underline{z = x^2 + y^2}$

as function of the others.]

$\dfrac{\partial w}{\partial x}$

$z = x^2 + y^2 \quad \Leftrightarrow \quad y^2 = z - x^2$

regard z as function of x, y. $\qquad$ regard y as funct of x, z.

Two Possible choice y, z.

☆ Since there may be ambiguity, we need better notations in place of $\frac{\partial w}{\partial x}$:

1° $\left(\frac{\partial w}{\partial x}\right)_y$: $\frac{\partial w}{\partial x}$ w/ x, y independent (i.e regard z as func of x, y)

2° $\left(\frac{\partial w}{\partial x}\right)_z$: $\frac{\partial w}{\partial x}$ w/ x, z independent (i.e regard y as function of x, z).

$$w = x^2 + y^2 + \underline{z}^2$$

for: 1°: $z = x^2 + y^2 \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = 2x$

$$\left(\frac{\partial w}{\partial x}\right)_y = 2x + 2\underset{\uparrow \atop x^2+y^2}{z}\left(\frac{\partial z}{\partial x}\right)_y = 2x + 2(x^2+y^2)\cdot 2x = 2x + 4x^3 + 4xy^2$$

for 2° $\left(\frac{\partial w}{\partial x}\right)_z = 2x + 2y\left(\frac{\partial y}{\partial x}\right)_z \longleftarrow y^2 = z - x^2 \Rightarrow 2y\cdot\frac{\partial y}{\partial x} = -2x \Leftrightarrow \frac{\partial y}{\partial x} = -\frac{2x}{2y}$

$$= 2x + 2y\cdot\left(-\frac{2x}{2y}\right)$$

$$= 2x - 2x$$

$$= 0$$

$*$ $\left(\frac{\partial w}{\partial x}\right)_y \boxed{\neq} \left(\frac{\partial w}{\partial x}\right)_z$

$w = x^2 + y^2 + \underline{z}^2$

for: 1°: $z = x^2 + y^2 \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = 2x$

$\boxed{\left(\frac{\partial w}{\partial x}\right)_y} = 2x + 2\underset{\substack{\uparrow \\ x^2+y^2}}{z} \left(\frac{\partial z}{\partial x}\right)_y = 2x + 2(x^2+y^2)\cdot 2x = \boxed{2x + 4x^3 + 4xy^2}$

$w = x^2 + y^2 + z^2 \quad w / z = x^2 + y^2 \Rightarrow w = x^2 + y^2 + \underset{\substack{\| \\ z}}{(x^2+y^2)^2} \Rightarrow \frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2$

for 2° $\boxed{\left(\frac{\partial w}{\partial x}\right)_z} = 2x + 2y\left(\frac{\partial y}{\partial x}\right)_z \longleftarrow y^2 = z - x^2 \Rightarrow 2y \cdot \frac{\partial y}{\partial x} = -2x \Leftrightarrow \frac{\partial y}{\partial x} = -\frac{2x}{2y}$

$= 2x + 2y \cdot \left(-\frac{2x}{2y}\right)$

$= 2x - 2x$

$\boxed{= 0}$

$w = x^2 + y^2 + z^2 = x^2 + (\overset{\substack{y^2 = z - x^2 \\ \|}}{z} - x^2) + z^2 = z + z^2 \Rightarrow \frac{\partial w}{\partial x} = 0$

More generally, for finding $\frac{\partial x_2}{\partial x_1}$ w/ ambiguity in k equ.

$F_{(i)}(x_1, \cdots x_n) = 0 \qquad 1 \le i \le k,$

$n \ge 2$, $1 \le k \le n-1$ we need to fix

- $(n-k)$ independent variables (including x_1)
- k dependent variables (including x_2)

Ex3: Find $\boxed{\left(\frac{\partial w}{\partial x}\right)_{y,z}}$ at $(x, y, z, t, w) = (0, \frac{1}{2}, \frac{1}{2}, 0, 0)$

if $\sin(w+\pi) = x + yzt$ $(*)$ and $t + e^{t^2} = x + y + z$ $(+)$

$\uparrow$

constrained equation.

Hint of Ex 3:

$\left(\frac{\partial}{\partial x}\right)_{y,z}$ on $(*)$ & $(\dagger)$

we have: $\cos(w+\pi)\left(\frac{\partial w}{\partial x}\right)_{y,z} = 1 + yz\left(\frac{\partial t}{\partial x}\right)_{y,z}$

$$(1+2te^{t^2})\left(\frac{\partial t}{\partial x}\right)_{y,z} = 1$$

Putting $(x,y,z,t,w) = (0,\frac{1}{2},\frac{1}{2},0,0)$ & Solving.

$$\Rightarrow \left(\frac{\partial w}{\partial x}\right)_{yz}\Big|_{(0,\frac{1}{2},\frac{1}{2})} = -\frac{5}{4}$$